Published by Institute of Physics Publishing for SISSA/ISAS
Received: *February 4, 2003*
Accepted: *March 30, 2003*

Quantum theory of noncommutative fields

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Abstract: Generalizing the noncommutative harmonic oscillator construction, we propose a new extension of quantum field theory based on the concept of "noncommutative fields". Our description permits to break the usual particle-antiparticle degeneracy at the dispersion relation level and introduces naturally an ultraviolet and an infrared cutoff. Phenomenological bounds for these new energy scales are given.

Keywords: Space-Time Symmetries, Beyond Standard Model.

Contents

1. Introduction

Relativistic quantum field theory (RQFT) is the general framework of our present microscopical theories. Its validity in particle physics has been confirmed by experiments covering a very wide range of energies, from the eV to the TeV [1]. From a modern perspective [2], RQFT is the necessary form adopted by any low-energy or "effective" theory satisfying the following three principles: special relativity, quantum mechanics and the cluster decomposition principle, which basically states that distant experiments yield uncorrelated results. The success of RQFT is then a confirmation of the validity of its ingredients, which describe correctly low-energy phenomena.

There seems however to be strong difficulties in obtaining a RQFT containing gravitation. Indeed there is no *a priori* reason why RQFT should be the correct framework of a high-energy theory, and even one or more of its ingredients could fail at these energies. In particular, Lorentz invariance could not be an exact symmetry at high energies, as recent developments in quantum gravity suggest [3, 4]. It is clear however that any high-energy theory of particle physics should reduce to a RQFT at low energies.

In this paper we propose an extension of the RQFT framework based on the notion of what we will call "a noncommutative field". Motivated by the appearance of noncommutative spaces in string theory [5], there has recently been quite a few developments on "noncommutative quantum mechanics" (NCQM), which is an extension of quantum mechanics (QM) consisting in the formulation of quantum mechanical systems on a noncommutative coordinate space (or even phase-space) [6, 7]. "Noncommutative quantum field theories", meaning quantum field theories on such spaces, have also been studied [8]. These theories violate relativistic invariance [9] and modify in a peculiar way the short-distance behavior of the theory (although there are still ultraviolet divergences).

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There is however another way of introducing a noncommutativity in quantum field theory. In NCQM, the coordinates (and momenta), which are the degrees of freedom of the system, are made noncommutative. The degrees of freedom in field theory are the fields at every point of space (and their conjugated momenta). Therefore the natural generalization of NCQM to field theory leads to a noncommutative field (instead of a field in a noncommutative space). In section 2 we show how a quantum theory of such a field can be constructed and then in section 3 we will study the properties of the extension of RQFT it provides. Finally in section 4 we will consider its phenomenological implications, the bounds that present experimental status produce on the size of the noncommutativity and the possibility to measure effects coming from this source.

2. The noncommutative field

2.1 Definition of the free noncommutative field theory

Let us consider a scalar field theory with two fields (ϕ_1, ϕ_2), i.e., a complex field $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$, and let us introduce a noncommutativity between the two fields at every point of space (for now we work in the Schrödinger picture where fields and momenta do not depend on time)

$$\left[\Phi(\boldsymbol{x}), \Phi^\dagger(\boldsymbol{x}')\right] = \theta\, \delta^3(\boldsymbol{x} - \boldsymbol{x}')\,. \tag{2.1}$$

We may consider at the same time a noncommutativity in the momenta $\Pi = (\pi_1 + i\pi_2)/\sqrt{2}$

$$\left[\Pi(\boldsymbol{x}), \Pi^\dagger(\boldsymbol{x}')\right] = B\, \delta^3(\boldsymbol{x} - \boldsymbol{x}')\,, \tag{2.2}$$

where θ and B are the parameters which parametrize the noncommutativity. The fields and their conjugated momenta are related by the conventional commutation relations

$$\left[\Phi(\boldsymbol{x}), \Pi^\dagger(\boldsymbol{x}')\right] = i\, \delta^3(\boldsymbol{x} - \boldsymbol{x}')\,, \tag{2.3a}$$

$$\left[\Phi^\dagger(\boldsymbol{x}), \Pi(\boldsymbol{x}')\right] = i\, \delta^3(\boldsymbol{x} - \boldsymbol{x}')\,. \tag{2.3b}$$

The free theory of the complex noncommutative quantum field is completely defined by the above commutation relations, together with the hamiltonian

$$\begin{aligned} H &= \int d^3\boldsymbol{x}\, \mathcal{H}(\boldsymbol{x})\,, \\ \mathcal{H}(\boldsymbol{x}) &= \Pi^\dagger(\boldsymbol{x})\Pi(\boldsymbol{x}) + \lambda \boldsymbol{\nabla}\Phi^\dagger(\boldsymbol{x})\boldsymbol{\nabla}\Phi(\boldsymbol{x}) + m^2\Phi^\dagger(\boldsymbol{x})\Phi(\boldsymbol{x})\,, \end{aligned} \tag{2.4}$$

where $\mathcal{H}$ is the hamiltonian density. Note that there is one additional dimensionless parameter λ in the hamiltonian as compared to the canonical theory because we already set the scale of the fields through the commutation relations (2.3).

The commutation relations (2.1) and (2.2) are not the most general ones to define a noncommutative field: one could introduce nonzero commutators between fields (and momenta) at different spatial points, which would then involve a much more complicated and arbitrary parametrization than that of eqs. (2.1) and (2.2). We will examine the properties and characteristic features of this simple definition for the noncommutative field in section 3.

In order to solve this theory, it is convenient to recall the method followed in standard RQFT ($\theta = B = 0$, $\lambda = 1$). There one identifies the "field" as a superposition of an infinite number of decoupled one-dimensional quantum harmonic oscillators, each with a frequency $\omega(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$, and then uses the solution of the harmonic oscillator in QM to define the Fock space of particle-states where the field acts. This suggests that considering the noncommutative generalization of the harmonic oscillator in QM might help us to define a Fock space for the noncommutative field.

2.2 Harmonic oscillator in noncommutative quantum mechanics

Let us consider a particle in two noncommuting spatial dimensions (the coordinates will be in correspondence with the two noncommuting real fields) in the presence of the harmonic oscillator potential (in correspondence with the relation between a free field theory and a superposition of oscillators) and a constant magnetic field [in correspondence with the noncommutativity in momenta eq. (2.2)].

This system is defined by the hamiltonian

$$H = \frac{\omega}{2}\left(\hat{p}_1^2 + \hat{p}_2^2 + \hat{q}_1^2 + \hat{q}_2^2\right) \tag{2.5}$$

and the commutation rules

$$[\hat{q}_1, \hat{q}_2] = i\,\hat{\theta}\,, \qquad [\hat{p}_1, \hat{p}_2] = i\,\hat{B}\,, \qquad [\hat{q}_i, \hat{p}_j] = i\,\delta_{ij}\,. \tag{2.6}$$

Note that we have expressed the hamiltonian and the commutation rules in terms of adimensional phase-space coordinates (hence the small angles over them) appropriately rescaled, and omitted $\hbar$ factors.

This problem was recently considered and solved in ref. [7]. Since we are interested in the case where the noncommutativity is going to be a small correction to RQFT, we will restrict in the following to the $\hat{B}\hat{\theta} < 1$ case. The appropriate way to solve this problem is to identify a linear transformation of the phase-space coordinates

$$\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2 \longrightarrow \hat{Q}_1, \hat{P}_1, \hat{Q}_2, \hat{P}_2\,, \tag{2.7}$$

so that the commutation rules in the new variables are

$$[\hat{Q}_1, \hat{Q}_2] = [\hat{P}_1, \hat{P}_2] = 0\,, \qquad [\hat{Q}_i, \hat{P}_j] = i\,\delta_{ij}\,, \tag{2.8}$$

and the hamiltonian in the new variables is still diagonal

$$H = \frac{\omega_1}{2}\left(\hat{P}_1^2 + \hat{Q}_1^2\right) + \frac{\omega_2}{2}\left(\hat{P}_2^2 + \hat{Q}_2^2\right). \tag{2.9}$$

However, there is no a unique linear transformation (2.7) satisfying eqs. (2.8) and (2.9). In fact the authors of ref. [7] got a very special (and, after having obtained the general result, complicated) linear transformation. In order to adequately solve our original problem (the free theory of the noncommutative field) it will prove convenient to work out the most general solution of this associated quantum mechanical problem. We will give now the final result.

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This system is equivalent to a set of two decoupled one-dimensional oscillators of frequencies[1]

$$\omega_1 = \omega \left[\sqrt{1 + \left(\frac{\hat{B} - \hat{\theta}}{2} \right)^2} + \left(\frac{\hat{B} + \hat{\theta}}{2} \right) \right], \tag{2.10a}$$

$$\omega_2 = \omega \left[\sqrt{1 + \left(\frac{\hat{B} - \hat{\theta}}{2} \right)^2} - \left(\frac{\hat{B} + \hat{\theta}}{2} \right) \right]. \tag{2.10b}$$

If we set $\hat{\theta} = \hat{B} = 0$ then $\omega_1 = \omega_2 = \omega$ and we recover the result of the symmetric bidimensional harmonic oscillator of frequency ω.

In order to have a simple expression for the most general linear transformation (2.7) which passes from the hamiltonian (2.5) and the commutation rules (2.6) to the hamiltonian (2.9) and the commutation rules (2.8) it is convenient to use the following combination of variables:

$$\begin{aligned} z &= \frac{\hat{q}_1 + i\hat{q}_2}{\sqrt{2}}, \qquad & w &= \frac{\hat{p}_1 + i\hat{p}_2}{\sqrt{2}}, \\ \bar{z} &= \frac{\hat{q}_1 - i\hat{q}_2}{\sqrt{2}}, \qquad & \bar{w} &= \frac{\hat{p}_1 - i\hat{p}_2}{\sqrt{2}}, \end{aligned} \tag{2.11}$$

instead of the original variables $\hat{q}_1$, $\hat{p}_1$, $\hat{q}_2$, $\hat{p}_2$. The creation-annihilation operators of the one-dimensional oscillators of frequencies ω_1, ω_2 are

$$\begin{aligned} a &= \frac{\hat{Q}_1 + i\hat{P}_1}{\sqrt{2}}, \qquad & b &= \frac{\hat{Q}_2 + i\hat{P}_2}{\sqrt{2}}, \\ a^\dagger &= \frac{\hat{Q}_1 - i\hat{P}_1}{\sqrt{2}}, \qquad & b^\dagger &= \frac{\hat{Q}_2 - i\hat{P}_2}{\sqrt{2}}. \end{aligned} \tag{2.12}$$

Then the most general linear transformation that allows to solve the problem of noncommutative quantum mechanics is

$$z = \eta\epsilon_1 \, e^{i\alpha} a + \epsilon_2 \, e^{i\beta} b^\dagger \,, \tag{2.13a}$$

$$\bar{z} = \eta\epsilon_1 \, e^{-i\alpha} a^\dagger + \epsilon_2 \, e^{-i\beta} b \,, \tag{2.13b}$$

$$w = -i\epsilon_1 \, e^{i\alpha} a + i\eta\epsilon_2 \, e^{i\beta} b^\dagger \,, \tag{2.13c}$$

$$\bar{w} = i\epsilon_1 \, e^{-i\alpha} a^\dagger - i\eta\epsilon_2 \, e^{-i\beta} b \,, \tag{2.13d}$$

where α, β are two angles which parametrize the most general linear transformation. Both angles appear only in exponential factors accompanying the a and b operators. Then we can use the freedom in the phase choice of the particle states to take, without any lost of generality, $\alpha = \beta = 0$ in eq. (2.13). The coefficients η, ϵ_1 and ϵ_2 are expressed in terms of

[1] This result can also be obtained directly from the second order equations for the operators $\hat{q}_i$ which one gets after eliminating $\hat{p}_i$ in the Hamilton equations $i\mathrm{d}\mathcal{O}/\mathrm{d}t = [\mathcal{O}, H]$, where $\mathcal{O}$ is a phase-space variable.

the noncommutative parameters

$$\eta = \sqrt{1 + \left(\frac{\hat{B} - \hat{\theta}}{2}\right)^2} - \left(\frac{\hat{B} - \hat{\theta}}{2}\right), \tag{2.14a}$$

$$\epsilon_1^2 = \frac{\hat{B} + \eta}{1 + \eta^2} = \frac{1/\eta + \hat{\theta}}{1 + \eta^2}, \tag{2.14b}$$

$$\epsilon_2^2 = \frac{\eta - \hat{\theta}}{1 + \eta^2} = \frac{1/\eta - \hat{B}}{1 + \eta^2}. \tag{2.14c}$$

Since there are only two parameters for the noncommutativity, we have a relation between these three coefficients

$$\epsilon_1^2 + \epsilon_2^2 = \frac{1}{\eta}. \tag{2.15}$$

Finally, we can obtain from eq. (2.14) the following simple relations:

$$\epsilon_1^2 - \eta^2 \epsilon_2^2 = \hat{B}, \qquad \eta^2 \epsilon_1^2 - \epsilon_2^2 = \hat{\theta}. \tag{2.16}$$

2.3 Construction of the noncommutative field

The noncommutative field is constructed from the above solution of the noncommutative quantum mechanics problem by considering an oscillator for each value of the momentum $\boldsymbol{p}$ of frequency $\omega(\boldsymbol{p}) = \sqrt{\lambda \boldsymbol{p}^2 + m^2}$. Then, the extension to the complex field Φ of the expression (2.13) of the z coordinate as a function of the creation and annihilation operators is

$$\Phi(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{\omega(\boldsymbol{p})}} \left[\eta(\boldsymbol{p})\epsilon_1(\boldsymbol{p})\, a_{\boldsymbol{p}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + \epsilon_2(\boldsymbol{p})\, b_{\boldsymbol{p}}^\dagger\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right], \tag{2.17}$$

which generalizes the conventional expression of the field in RQFT as a superposition of infinite oscillators (each for every momentum) at every space point. Eq. (2.17) includes the plane-wave factors $e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$, an explicit momentum-dependence in the coefficients η, ϵ_1 and ϵ_2, and a global factor $1/\sqrt{\omega}$ to take into account the rescaling between the adimensional coordinates and the field. The momentum has the analogue extension of the expression of w in eq. (2.13)

$$\Pi(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \sqrt{\omega(\boldsymbol{p})} \left[-i\, \epsilon_1(\boldsymbol{p})\, a_{\boldsymbol{p}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + i\, \eta(\boldsymbol{p})\epsilon_2(\boldsymbol{p})\, b_{\boldsymbol{p}}^\dagger\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right]. \tag{2.18}$$

Eqs. (2.17) and (2.18) (and their respective conjugated expressions) give the fields and momenta as a function of creation and annihilation operators. If these satisfy the commutation rules

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}^\dagger\right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}'), \tag{2.19a}$$

$$\left[b_{\boldsymbol{p}}, b_{\boldsymbol{p}'}^\dagger\right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}'), \tag{2.19b}$$

and we choose the noncommutative parameters of the system of quantum mechanics corresponding to each momentum as

$$\hat{\theta}(\boldsymbol{p}) = \theta\, \omega(\boldsymbol{p}), \qquad \hat{B}(\boldsymbol{p}) = \frac{B}{\omega(\boldsymbol{p})}, \tag{2.20}$$

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with θ, B constants which do not depend of the momentum, then the calculation of the commutators of fields and momenta give the commutation rules (2.1), (2.2) and (2.3). This proves that the field constructed as in eq. (2.17) is a representation of the noncommutative field in the Fock space defined by the creation and annihilation operators $a_{\boldsymbol{p}}$, $a^\dagger_{\boldsymbol{p}}$, $b_{\boldsymbol{p}}$ and $b^\dagger_{\boldsymbol{p}}$.

Moreover using the representation of fields and momenta as a linear combination of creation and annihilation operators, eqs. (2.17) and (2.18), one can express the hamiltonian (2.4) in the form

$$H = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \left[E_1(\boldsymbol{p}) \left(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + \frac{1}{2} \right) + E_2(\boldsymbol{p}) \left(b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}} + \frac{1}{2} \right) \right], \tag{2.21}$$

which shows that the theory of the free complex noncommutative field is a theory of free particles of two types. $E_1(\boldsymbol{p})$ and $E_2(\boldsymbol{p})$ give the expressions of the energy of one of these particles with momentum $\boldsymbol{p}$. These energies are simply the frequencies in eq. (2.10) of the two decoupled oscillators appearing in the solution of the quantum mechanical system corresponding to each momentum $\boldsymbol{p}$ with the parameters of noncommutativity given in eq. (2.20)

$$E_1(\boldsymbol{p}) = \omega(\boldsymbol{p}) \left[\sqrt{1 + \frac{1}{4}\left(\frac{B}{\omega(\boldsymbol{p})} - \theta\omega(\boldsymbol{p}) \right)^2} + \frac{1}{2}\left(\frac{B}{\omega(\boldsymbol{p})} + \theta\omega(\boldsymbol{p}) \right) \right], \tag{2.22a}$$

$$E_2(\boldsymbol{p}) = \omega(\boldsymbol{p}) \left[\sqrt{1 + \frac{1}{4}\left(\frac{B}{\omega(\boldsymbol{p})} - \theta\omega(\boldsymbol{p}) \right)^2} - \frac{1}{2}\left(\frac{B}{\omega(\boldsymbol{p})} + \theta\omega(\boldsymbol{p}) \right) \right]. \tag{2.22b}$$

In summary we have seen that the free theory of the scalar noncommutative field can be solved in a similar way as in the conventional case of RQFT. It is a theory of free particles. The simplest way to incorporate interactions is by using the same hamiltonians as in RQFT, now in terms of noncommutative fields. Then the solution of the free theory can be taken as a starting point for a perturbative treatment of interactions analogously to what is done in RQFT: identification of propagators, Feynman rules, etc. We will sketch this procedure in the following section, and examine the characteristic properties of the noncommutative theories defined in such a way.

3. Properties of the quantum theory of noncommutative fields

The free noncommutative field was defined by the commutation relations (2.1), (2.2) and by the hamiltonian (2.4). This simple implementation of a noncommutativity in field space has the following properties:

1. Standard RQFT is trivially recovered in the $\theta \to 0$, $B \to 0$, $\lambda \to 1$ limit. As in the standard case, the quantum theory is obtained from a classical hamiltonian which is relativistic invariant [the apparent noninvariance of $\lambda \neq 1$ in eq. (2.4) is fictitious; as we remarked before, it is a consequence of the choice of the scale for the fields in order to write the standard commutation relations between fields and

momenta eqs. (2.3)], but now we follow a quantification procedure, given by the new commutation relations (2.1) and (2.2), which explicitely violates Lorentz symmetry. This is what we understand by the "quantum theory of a noncommutative field".

2. The hamiltonian density defined in eq. (2.4) is made of fields which commute at different space points, and therefore satisfies

$$[\mathcal{H}(\boldsymbol{x}), \mathcal{H}(\boldsymbol{x}')] = 0 \qquad \text{for} \quad \boldsymbol{x} \neq \boldsymbol{x}' \,. \tag{3.1}$$

This property is essential in RQFT to guarantee that the S-matrix will be Lorentz-invariant. More specifically, what is required in RQFT is that

$$[\mathcal{H}(x), \mathcal{H}(x')] = 0 \qquad \text{for} \quad (x - x')^2 \geq 0 \,, \tag{3.2}$$

which is equivalent to eq. (3.1) in a Lorentz-invariant theory. The two conditions are not equivalent however when relativistic invariance is lost. In fact, in the quantum theory of the noncommutative field, the hamiltonian density satisfies eq. (3.1) but not eq. (3.2). The preservation of the property (3.1) is in any case welcome since it allows to speak consistently about the concept of a hamiltonian density. Without this condition, the energy of a closed finite system could depend on the energy of another system very far away. Commutation relations more general than eqs. (2.1) and (2.2) would violate eq. (3.1).

3. Keeping the property (3.1) requires the introduction of two real fields as the only way to implement a noncommutativity in field space. This leads to a theory with two types of particles which correspond to the particle and the antiparticle in the $\theta, B \to 0, \lambda \to 1$ limit (conventional RQFT). Particle and antiparticle are no longer degenerated in this extension of RQFT, and their energy is different from the standard expression $\sqrt{\boldsymbol{p}^2 + m^2}$ by small corrections parametrized by θ, B and λ. The theory naturally incorporates in this way a matter-antimatter asymmetry.

4. We have a new, specific form of the dispersion relation, or relation between energy and momentum of a particle, eq. (2.22), which is no longer Lorentz-invariant, while the theory still preserves rotational symmetry. Relativistic invariance is therefore an ingredient which is lost in this extension of RQFT. We will see in section 4 that this symmetry is violated not only at high energies, but, surprisingly enough, also at low energies, being still compatible with phenomenological observations. Relativistic causality is also violated, as we check later in this section.

5. An essential property of RQFT, which in principle should hold in any sensible physical theory, is the cluster decomposition principle: experiments which are sufficiently separated in space should have unrelated results. A general theorem states that the S-matrix satisfies this crucial requirement if the hamiltonian can be expressed as a sum of products of creation and annihilation operators, with suitable non-singular coefficients [2]. This theorem garantees that the cluster property still holds in the noncommutative extension of RQFT.

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To discuss causality and the formulation of perturbation theory we have to consider the field operator in the interaction picture

$$\begin{aligned}\Phi(\boldsymbol{x},t) &= e^{iH_0 t}\,\Phi(\boldsymbol{x})\,e^{-iH_0 t} \\ &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{\omega(\boldsymbol{p})}}\left[\eta(\boldsymbol{p})\epsilon_1(\boldsymbol{p})\,a_{\boldsymbol{p}}\,e^{-iE_1(\boldsymbol{p})t}e^{i\boldsymbol{p}.\boldsymbol{x}}+\epsilon_2(\boldsymbol{p})\,b^\dagger_{\boldsymbol{p}}\,e^{iE_2(\boldsymbol{p})t}e^{-i\boldsymbol{p}.\boldsymbol{x}}\right].\end{aligned} \tag{3.3}$$

From eq. (3.3) one can calculate the commutator of operators at different times

$$\left[\Phi(\boldsymbol{x},t),\Phi^\dagger\left(\boldsymbol{x}',t'\right)\right] \tag{3.4}$$

and verify that it is different from zero at causally disconnected points ($(\boldsymbol{x}-\boldsymbol{x}')^2 > (t-t')^2$) owing to the noncommutativity.

The modification to the standard propagator caused by the noncommutativity is rather simple. One has

$$\langle 0|\mathcal{T}(\Phi(\boldsymbol{x},t)\Phi^\dagger(\boldsymbol{x}',t')|0\rangle = \int \frac{d^4p}{(2\pi)^4}\,e^{-ip^0(t-t')}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\frac{i\,(1-\theta B+\theta p^0)}{(p^0-E_1(\boldsymbol{p})+i\epsilon)(p^0+E_2(\boldsymbol{p})-i\epsilon)}, \tag{3.5}$$

that is, the effect of the noncommutativity is a displacement in the position of the poles

$$\pm\sqrt{\boldsymbol{p}^2+m^2}\;\to E_1(\boldsymbol{p}),-E_2(\boldsymbol{p}), \tag{3.6}$$

together with a modification of the residues

$$\pm\frac{1}{2\sqrt{\boldsymbol{p}^2+m^2}}\to\frac{(1-\theta B+\theta E_1(\boldsymbol{p}))}{(E_1(\boldsymbol{p})+E_2(\boldsymbol{p}))},-\frac{(1-\theta B-\theta E_2(\boldsymbol{p}))}{(E_1(\boldsymbol{p})+E_2(\boldsymbol{p}))}. \tag{3.7}$$

We will finally note that there is not any obstruction to the introduction of gauge symmetries in the theory of noncommutative fields. In the free noncommutative complex scalar field theory we have a global U(1) symmetry that can be made local in a hamiltonian with interaction terms containing so many Φ fields as $\Phi^\dagger$ fields, if every derivative of the field appears in the combination $-i\,\boldsymbol{\nabla}\Phi+\boldsymbol{A}\Phi$. We leave further exploration on the dynamics of interactions in theories of noncommutative fields for future work and consider in the following section the phenomenological implications coming from the solution of the free theory.

4. Phenomenological bounds on the parameters of noncommutativity

We would like to show in this section how the quantum theory of noncommutative fields can be a sensible extension of RQFT, in the sense that it does not contradict the present understanding of low-energy phenomena and, at the same time, may have observable consequences.

The effect of the noncommutativity at the level of the free theory is the substitution of the particle or antiparticle states of momentum $\boldsymbol{p}$ and energy $E=\sqrt{\boldsymbol{p}^2+m^2}$ in RQFT by two states of energies $E_1(\boldsymbol{p})$ and $E_2(\boldsymbol{p})$ given by eqs. (2.22). From these expressions one sees that if

$$B \ll \sqrt{\lambda\boldsymbol{p}^2+m^2}\;\ll\;1/\theta \Rightarrow E_1(\boldsymbol{p})\approx E_2(\boldsymbol{p})\approx\sqrt{\lambda\boldsymbol{p}^2+m^2} \tag{4.1}$$

and the relativistic dispersion relation is recovered in the $\lambda=1$ limit.

We then have two energy scales coming from the noncommutativity: B and $1/\theta$, and from the dispersion relation we see that these two scales are respectively the infrared (IR) and ultraviolet (UV) scales which limit the range of validity of the relativistic invariant theory.

The noncommutativity induces a violation of relativistic invariance both at high and low energies. A maximum velocity of propagation different from the speed of light ($\lambda \neq 1$) and high-energy violations coming from the presence of an UV scale are properties which have been explored in other contexts [1, 3, 4, 10, 11], especially those trying to incorporate effects coming from the Planck length. What is new in the present extension of RQFT is the presence of an additional IR scale and a violation of relativistic invariance at low energies.

This result allows to explore consequences of the noncommutativity already at the level of the kinematics, without a necessity of considering in detail the dynamics. The analysis will alternatively give restrictions on the values of the parameters of the noncommutativity, based on the success of the RQFT description of nature. We note here that this kinematics study needs a further assumption. The new dispersion relations (2.22) were obtained for the free theory of the scalar noncommutative field. The extension to a free theory of fermions is not however a trivial task, and we leave it for future work. We will now make the phenomenological analysis to obtain bounds on the parameters of noncommutativity assuming that a similar dispersion relation holds for fermions.

The most sensitive experiments to detect a violation of relativistic invariance at high energies are those involving ultra high-energy cosmic rays (UHECR). Not only they reach energies as high as 10^{20} eV, but they are even sensitive to effects parametrized by much larger energy scales, such as the Planck scale, thanks to amplification mechanisms coming from the presence of very different scales. This is what happens at the very end of the cosmic ray spectrum: relativistic kinematics predicts a cutoff in the spectrum for UHECR coming from distant sources (the GZK cutoff) [13], caused by the energy loss they experiment in their interaction with the cosmic background radiation (CBR), which seems to be avoided somehow [14]. Relativistic invariance violations induced by the Planck scale are, among others [15], a possible explanation for the disappearance of the GZK cutoff [3, 11, 12, 16]. The sensitivity to the Plank scale results in this case from the presence of a very small energy scale: the kinetic energy of photons of the CBR, 10^{-3} eV.

In this range of momenta, we expect $|\theta|\,(\boldsymbol{p}^2 + m^2) \gg |B|$, and then B can be neglected in the dispersion relations (2.22). Bounds on $1/\theta$ in the $B \to 0$ limit coming from the physics of the UHECR were studied in ref. [12]. Taking the energy-momentum relation eq. (2.22a) for the particle, and eq. (2.22b) for the antiparticle, so that the noncommutative field eq. (2.17) generalizes the expression of the conventional field in RQFT, as a linear combination of particle annihilation operators $a_{\boldsymbol{p}}$ and antiparticle creation operators $b^\dagger_{\boldsymbol{p}}$, then the analysis of ref. [12] shows that the sign of θ has to be negative. This is because $E_1(\boldsymbol{p})$ with $\theta > 0$ would generate a mechanism of energy loss for particles independent of the CBR: particle disintegrations prohibited by relativistic kinematics, would be now allowed. The observation of UHECR excludes then this possibility. With $\theta < 0$, $E_1(\boldsymbol{p}) < \omega(\boldsymbol{p})$, and the interaction with the CBR is now kinematically forbidden so that the GZK cutoff no longer exists. Experiments coming in the near future [17] will clarify the situation with respect to the GZK cutoff violation, which will then be a stringent test for the theory of noncommutative fields. The bounds provided for the UV energy scale are [12]: 10^{21} eV $\lesssim 1/|\theta| \lesssim 10^{43}$ eV.

Let us now consider the IR corrections to the relativistic dispersion relation given by eqs. (2.22). In the range of momenta where $|\theta|\,(\boldsymbol{p}^2+m^2) \ll |B|$, we can neglect in a first approximation the effects parametrized by the UV scale $(1/\theta)$ and then we obtain

$$E_1(\boldsymbol{p}) \approx \sqrt{\boldsymbol{p}^2+m^2+\frac{B^2}{4}}-\frac{B}{2} \tag{4.2}$$

for the energy of the particle, and an analogous expression for the antiparticle (replacing $-B/2$ by $+B/2$). We then see that the effect of the noncommutativity is a constant contribution to the energy (opposite in sign for particles and antiparticles) and a "renormalization" of the mass $m^2 \to m^2_{\text{eff}} = m^2 + B^2/4$. Conservation laws in physical process will however make invisible the $\pm B/2$ constant contributions to the energy coming from the noncommutativity. On the other hand, the bound $B^2/4 \leq m^2_{\text{eff}}$ will restrict the value of B from the bounds to neutrino masses. The β disintegration of tritium, which is the most sensitive experiment to neutrino mass [18], gives $B < 5$ eV.

Finally, bounds on the adimensional parameter λ were considered in ref. [11] in different scenarios. The typical bound is $|1-\lambda| \leq 10^{-23}$. A more detailed phenomenological analysis including cosmological implications of the noncommutativity will be given elsewhere.

Acknowledgments

We would like to thank J. Clemente-Gallardo for discussions. This work has been partially supported by the grants 1010596, 7010596 and 3000005 from Fondecyt-Chile, by M.AA.EE./AECI and by MCYT (Spain), grants FPA2000-1252 and FPA2001-1813.

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